Complex Geometry Exercises

Week 4

Exercise 1. *Prove that, up to isomorphism, we have the following correspondences:*

- *real vector bundles of rank* $r \xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{C}^\infty(X,\mathbb{R})\big)$,
- *complex vector bundles of rank* $r \xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{C}^\infty(X,\mathbb{C})\big)$
- *holomorphic vector bundles of rank* $r \xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{O}_X)\big)$,

where $\mathrm{GL}(r, \mathcal{F})$ *is the sheaf of invertible rank* k *matrices with coefficients in the sheaf* $\mathcal{F}$.

Exercise 2. *Show that any short exact sequence of holomorphic vector bundles*

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0\ ,$$

where L *is a line bundle, induces the short exact sequence*

$$0 \longrightarrow L \otimes \Lambda^{k-1} E \longrightarrow \Lambda^k E \longrightarrow \Lambda^k F \longrightarrow 0.$$

Exercise 3. *Let* L_1 *and* L_2 *be holomorphic line bundles such that* $L_1|_{X\setminus Y} = L_2|_{X\setminus Y}$ *for* $Y \subset X$ *a complex submanifold of codimension* ≥ 2. *Show that* $L_1 \cong L_2$.

Exercise 4. *Prove that for a 1-dimensional connected complex manifold* X,

$$K(X) \cong \left\{ f : X \to \mathbb{P}^1 \,\middle|\, f \text{ is holomorphic, } f \not\equiv \infty \right\}.$$

Exercise 5. *Let* $X = \mathbb{C}^n/\Lambda$ *be a complex torus. Show that:*

(a) $\tau_X = \mathcal{O}_X^n$, $K_X = \mathcal{O}_X$

(b) *kod*$(X) = 0$ *and kod*$(Y) \geq 0$ *for every (positive-dimensional) submanifold* $Y \subset X$

(continues on the back)

Exercise 6 (Veronese embedding). *Show that the map defined by global sections $H^0(\mathbb{P}^1, \mathcal{O}(2))$ is an embedding*

$$\phi : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$

realising $\mathbb{P}^1$ as the zero set of a homogeneous polynomial of degree 2.

Exercise 7. *Assume X is a connected complex manifold.*

(i) Show that a holomorphic line bundle L over X is isomorphic to the trivial bundle if and only if both L and L^* admit a non-trivial global section.

(ii) Deduce $H^0(\mathbb{P}^n, \mathcal{O}(k)) = 0$ for $k < 0$.

(iii) Prove (or convince yourself) that there exist line bundles such that

$$H^0(X, L) = 0 = H^0(X, L^*) \ .$$